Note Title 01-11-2016

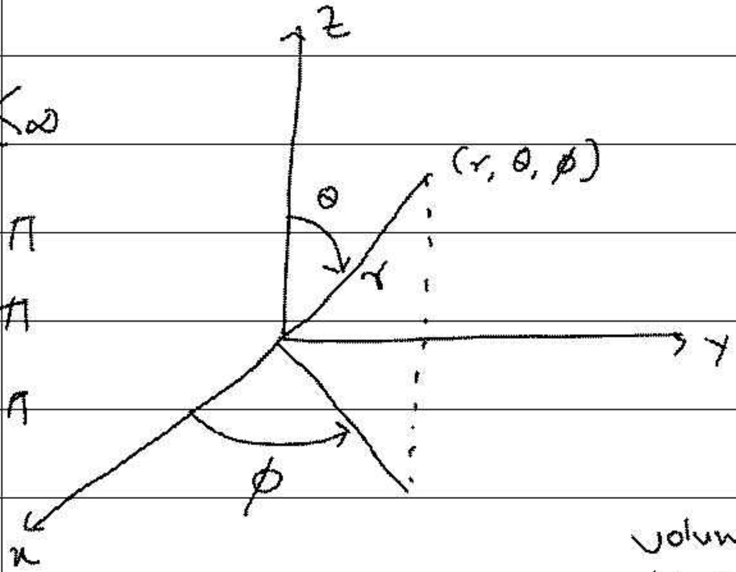
Rotational Motion

$0 \leq r \leq \infty$

~~$\theta \leq \pi$~~

$0 \leq \theta \leq \pi$

$0 \leq \phi \leq 2\pi$



$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

volume element

$d\tau = dx\,dy\,dz$ cartesian

$d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi$

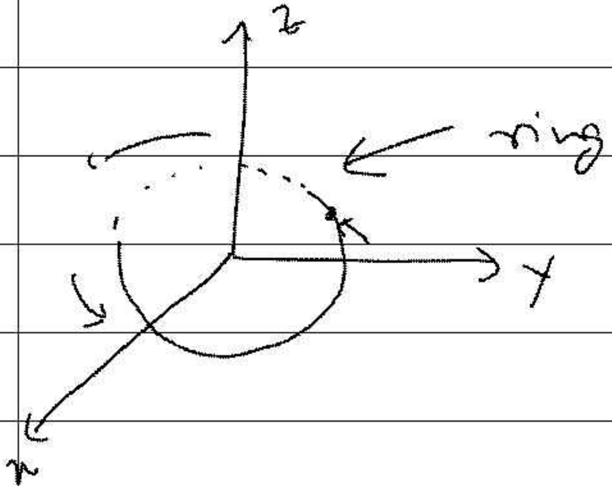
$d\tau = \underbrace{r^2 \sin\theta\, dr\, d\theta\, d\phi}_{\text{spherical polar coordinates}}$

$$-\frac{\hbar^2}{2m}\underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)}_{\nabla^2 \text{ Laplacian}}\psi + V\psi = E\psi(x, y, z) \quad \dots (1)$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi \quad \dots (2)$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \quad \dots (3)$$

Particle on a ring!



'r' is fixed
xy-plane
V = 0

$$\nabla^2 = \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \phi^2} \quad \cdots \quad (4)$$

$$-\frac{\hbar^2}{2m} \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2} = E \psi(\phi) \quad \cdot \quad (5)$$

$$-\frac{\hbar^2}{2mr^2} \frac{1}{\sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2} = E \psi(\phi)$$

$\theta = \pi/2 \quad \sin\theta = 1$

$$-\frac{\hbar^2}{2mr^2} \frac{\partial^2 \psi}{\partial \phi^2} = E \psi(\phi)$$

$$-\frac{\hbar^2}{2I} \frac{\partial^2 \psi}{\partial \phi^2} = E \psi(\phi) \quad \cdots \quad (6)$$

$$\psi(\phi) = a_{m_l} e^{i m_l \phi} \quad \cdots \quad (7)$$

general sol.n of (6)

$$\Phi(\phi) = a_{m_l} e^{i m_l \phi} \quad \cdots \quad (8)$$

$$-\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\phi^2}\left(a_{m_l}e^{im_l\phi}\right)$$

$$= -\frac{\hbar^2}{2I}(im_l)^2\,a_{m_l}e^{im_l\phi}$$

$$= \underbrace{\frac{m_l^2\hbar^2}{2I}}_{E}\,\Phi(\phi)$$

$$E = \frac{m_l^2\hbar^2}{2I} \quad \cdots \quad (9)$$

$$\Phi(\phi) = a_{m_l}e^{im_l\phi}$$

$$a_{m_l}e^{im_l(0+2\pi)} = a_{m_l}e^{im_l 0}$$

$$e^{im_l 2\pi} = e^{im_l 0} = 1$$

$$e^{im_l 2\pi} = 1$$

$$m_l = 0, \pm 1, \pm 2, \pm 3, \pm 4 \cdots\cdots$$

quantum number

$$\int_0^{2\pi}\Phi^*(\phi)\,\Phi(\phi)\,d\phi = 1$$

$$\int_0^{2\pi} a_{m_l}^* e^{-im_l\phi} a_{m_l} e^{im_l\phi} \, d\phi = 1$$

$$|a_{m_l}|^2 \int_0^{2\pi} e^{-im_l\phi} e^{im_l\phi} \, d\phi = 1$$

$$a_{m_l}^2 = \frac{1}{2\pi}$$

Normalisation constant $\rightarrow$ $a_{m_l} = \frac{1}{\sqrt{2\pi}}$

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}} e^{im_l\phi}$$ $\leftarrow$ full form of wave fn. (10)

Angular Momentum

$$L = r \times p$$

$$L = \begin{vmatrix} i & j & k \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$= i \underbrace{(yp_z - zp_y)}_{\hat{l}_x} - j \overbrace{(xp_z - zp_x)}^{-\hat{l}_y} + k \underbrace{(xp_y - yp_x)}_{\hat{l}_z}$$

$\hat{\ell} = i\hat{\ell}_x + j\hat{\ell}_y + k\hat{\ell}_z$

$$\left.\begin{aligned} \hat{\ell}_x &= y p_z - z p_y \\ \hat{\ell}_y &= z p_x - x p_z \\ \hat{\ell}_z &= x p_y - y p_x \end{aligned}\right\} \text{angular momentum components}$$

$$\hat{\ell}_z = x\left(-i\hbar\frac{\partial}{\partial y}\right) - y\left(-i\hbar\frac{\partial}{\partial x}\right)$$

$$\underline{\hat{\ell}_z = -i\hbar\frac{\partial}{\partial\phi}} \qquad \left[\text{compare } \hat{p}_x = -i\hbar\frac{\partial}{\partial x}\right]$$

$$\hat{\ell}_z \frac{1}{\sqrt{2\pi}} e^{im_\ell\phi} = -i\hbar\frac{\partial}{\partial\phi}\frac{1}{\sqrt{2\pi}} e^{im_\ell\phi}$$

$$= m_\ell\hbar \frac{1}{\sqrt{2\pi}} e^{im_\ell\phi}$$

$\Phi(\phi)$ is an eigenfunction of $\hat{\ell}_z$ with the eigenvalue $\underline{\underline{m_\ell\hbar}}$

<u>Particle on a sphere</u> $\rightarrow \theta, \phi$